PARTICLES OF GENERALIZED STATISTICS, QUANTUM LOGIC AND CATEGORIES *†

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Abstract

A collection of brilliant and original unfinished ideas by Wladyslaw Marcinek (1952-2003) in particle interactions, categorical approach to generalized statistics, qubits and quantum logic, entwined operators, cobordisms and noncommutative Fock space.

KEYWORDS : quantum statistics, composite state, qubit, quantum logic, Yang-Baxter equation, entwining structure, crossed product, graph, cobordism

*These are last unfinished ideas by Władysław Marcinek who unexpectedly passed away on June 9, 2003. Prepared for publication by Steven Duplij.

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1 Composite Systems and Regularity [1]

Let us assume that an interacting system of charged particles and quanta is given as a starting point for the study of particle processes. It is natural to assume that there is a set of initial configurations of the system and there is also a set of final configurations representing possible results of interactions. An arbitrary initial configuration can be transform into final one as a result of a sequences of vertex interactions. Every such sequence is said to be a *process*. The proper physical meaning have the initial and final configurations, intermediate steps are virtual. A process with an initial configuration and one final configuration and without intermediate steps is said to be *primary*. An arbitrary process can be built from primary ones. If the time evolution of the system can be described in an unique way as a sequence of transformations of initial configuration, then we say that the system is *consistent* and equipped with a *generalized quantum history*. In this approach to quantum theory the whole universe is represented by a class of 'histories'. In this formalism the standard Hamiltonian time-evolution is replaced by a partial semigroup called a 'temporal support' [1].

It is natural to assume that the whole world is divided into two parts: a classical particle system and its quantum environment. The classical system represents an observed reality, this particles which really exist and can be detect. The quantum environment represents all quantum possibilities to become a part of the reality in the future [2].

An algebraic model of an interacting system of charged particles and quanta of an external field has been developed by the author [3–5]. The model is based on the assumption that a given charged particle is transform under interaction into a composite system consisting a charge and quanta of the quantum field. The system containing a particle dressed with a single quantum of external field is said to be a *quasiparticle*. Quasiparticles behave like free particles equipped with generalized statistics. In this paper we develop the algebraic model considered in previous publications for systems with generalized statistics [3, 6, 7]. We use the category theory in order to construct the model. The initial data for construction of category $\mathfrak{C}$ relevant for our model is described. In our approach objects of the category $\mathfrak{C}$ represent physical objects like charged particles, quasiparticles, quasiholes, different species of quanta of an external field, etc... Functors or multifunctors represent interactions, creations and annihilations, etc... There is a collection of (multi)-functors representing primary act of interactions. Starting from

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the collection we can construct a collection of higher - ary functors representing multiquasiparticle processes. The problem is to describe a collection of natural isomorphisms such that our constructions are unique up to these isomorphisms.

Let us consider a system of charged particles interacting with a quantum environment. States of the system represent all initial configurations. These initial configurations are transformed into final configuration as a result of interaction of the system with the quantum environment. In this transformation every charged particle is converted into a composite system consisting a few charges and quanta from the quantum environment. Such a transformation is said to be a composition. A conversion of a composite system into his parts is called a decomposition. Note that both transformations, a composition and a decomposition need not to be invertible but they can be regular.

A system which contains a charge and certain number of quanta as a result of interaction with the quantum field is said to be a *dressed particle* [5]. Next we assume that every dressed particle is a composite object equipped with an internal structure. Obviously the structure of dressed particles is determined by the interaction with the quantum field. We describe a dressed particle as a non-local system which contains n centers (vertexes). Two systems with n and m centers, respectively, can be 'composed' into one system with $n + m$ centers. All centers as members of a given system behave like free particles moving on certain effective space $\mathcal{M}$. Every center is also equipped with ability for absorption and emission of quanta of the quantum field. Assume that there are quanta of N different kindS. A centre dressed with a single quantum of the field is said to be a *quasiparticle*. In this way N is not a number of real particles but a number of quasparticles. In our approach a center equipped with two quanta forms a system of two quasiparticles.

A center with an empty place for a single quantum is said to be a *quasihole*. A centre which contains any quantum is said to be *neutral*. A neutral center can be transform into a quasiparticle or a quasihole by an absorption or emission process of single quantum, respectively. In this way the process of absorption of quanta of quantum field by a charged particle is equivalent to a creation of quasiparticles and emission provide to annihilation of quasiparticles. Note that there is also the process of mutual annihilation of quasiparticles and quasiholes.

2 Generalized Qubits and Quantum Statistics [2]

In the last years a few different approaches to quantum statistics which generalize the usual boson or fermion statistics has been intensively developed by several authors. The so–called q–statistics and corresponding q–relations have been studied by Greenberg [8, 9], Mohapatra [10], Fivel [11] and many others, see [12–14] for example. The deformation of commutation relations for bosons and fermions corresponding to quantum groups $SU_q(2)$ has been given by Pusz and Woronowicz [15, 16]. The q–relations corresponding to superparticles has been considered by Chaichian, Kulisch and Lukierski [17]. Quantum deformations have been also studied by Vokos [18], Fairle and Zachos [19] and many others.

It is interesting that in the last years new and highly organized structures of matter has been discovered. For example in fractional quantum Hall effect a system with well defined internal order has appears [50]. Another interesting structures appear in the so called $\frac{1}{2}$ electronic magnetotransport anomaly [51, 52], high temperature superconductors or laser excitations of electrons. In these cases certain anomalous behaviour of electron have appear. The existence of new ordered structures depends on the existence of certain specific additional excitations. Hence we can restrict our attention to the study of possibility of appearance for these excitations. For the description of such possible excited states we use the concept of dressed particles. We assume that every charged particle is equipped with ability to absorb quanta of the external field. A system which contains a particle and certain number of quanta as a result of interaction with the external field is said to be dressed particle. A particle without quantum is called undressed or a quasihole. The particle dressed with two quanta of certain species is understand as a system of two new objects called quasiparticles. A quasiparticle is in fact the charged particle dressed with a single quantum. Two quasiparticles are said to be identical if they are dressed with quanta of the same species. In the opposite case when the particle is equipped with two different species of quanta then we have different quasiparticles. We describe excited states as composition of quasiparticles and quasiholes. It is interesting that quasiparticles and quasiholes have also their own statistics. We give the following assumption for the algebraic description of excitation spectrum of single dressed particle.

Fundamental Assumptions:

Assumption 0: The ground state. There is a state $|0 >= \mathbf{1}$ called the ground one. There is also the conjugate ground state $< 0| \equiv \mathbf{1}^*$. This is the state of the

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system before intersection.

Assumption 1: Elementary states. There is an ordered (finite) set of single quasiparticle states $S := \{x^i : i = 1, \ldots, N < \infty\}$. These states are said to be elementary (simple). They represent elementary excitations of the system. We assume that the set S of elementary states forms a basis for a finite linear space E over a field of complex numbers $\mathbb{C}$.

Assumption 2: Elementary conjugated states. There is also a corresponding set of single quasihole states $S^* := \{x^{*i} : i = N, N-1, \ldots, 1\}$. These states are said to be conjugated. The set S^* of conjugate states forms a basis for the complex conjugate space E^*. The pairing $(.|.) : E^* \otimes E \longrightarrow \mathbb{C}$ is given by $(x^{*i}|x^j) := \delta^{ij}$.

Assumption 3: Composite states. There is a set of projectors $\Pi_n : E^{\otimes n} \longrightarrow E^{\otimes n}$ such that we have a n-multilinear mapping $\odot_n : E^{\times n} \longrightarrow E^{\otimes n}$ defined by the following formula $x^{i_1} \odot \cdots \odot x^{i_n} := \Pi_n(x^{i_1} \otimes \cdots \otimes x^{i_n})$. The set of n–multiquasiparticle states is denoted by $P^n(S)$. All such states are result of composition (or clustering) of elementary ones. These states are also called composite states of order n. They represent additional excitations charged particle under interaction. In this way for multiquasiparticle states we have the following set of states $P^n(S) := \{x^\sigma \equiv x^{i_1} \odot \cdots \odot x^{i_n} : \sigma = (i_1, \ldots, i_n) \in I\}$. Here I is a set of sequences of indices such that the above set of states forms a basis for a linear space $\mathcal{A}^n$. We have $\mathcal{A}^n = Im(\Pi_n)$. Obviously we have where $\mathcal{A}^0 \equiv \mathbf{1}\mathbb{C}$, $\mathcal{A}^1 \equiv E$ and $\mathcal{A}^n \subset E^{\otimes n}$.

Assumption 4: Composite conjugated states. We also have a set of projectors $\Pi^*_n : E^{*\otimes n} \longrightarrow E^{*\otimes n}$ and the corresponding set of composite conjugated states of length n

$$P^n(S^*) := \{x^{*\sigma} \equiv x^{*i_n} \odot \cdots \odot x^{*i_1} : \sigma = (i_1, \ldots, i_n) \in I\}. \tag{1}$$

The set $P^n(S^*)$ of composite conjugated states of length n forms a basis for a linear space $\mathcal{A}^{*n}$.

Assumption 5: Algebra of states. The set of all composite states of arbitrary length is denoted by $P(S)$. For this set of states we have the following linear space $\mathcal{A} := \bigoplus_n \mathcal{A}^n$. If the formula

$$m(s \otimes t)s \equiv s \odot t := \Pi_{m+n}(\tilde{s} \otimes \tilde{t}) \tag{2}$$

for $s = \Pi_m(\tilde{s}), t = \Pi_n(\tilde{t})$, $\tilde{s} \in E^{\otimes n}$, $\tilde{t} \in E^{\otimes m}$, defines an associative multiplication in $\mathcal{A}$, then we say that we have an algebra of states. This algebra represents excitation spectrum for single dressed particle.

Assumption 6: Algebra of conjugated states. The set of composite conjugated states of arbitrary length is denoted by $P(S^*)$. We have here a linear space $\mathcal{A}^* := \bigoplus_n \mathcal{A}^{*n}$. If m is the multiplication in $\mathcal{A}$, then the multiplication in $\mathcal{A}^*$ corresponds to the opposite multiplication $m^{op}(t^* \otimes s^*) = (m(s \otimes t))^*$.

We define creation operators for our model as multiplication in the algebra $\mathcal{A}$ as $a_s^+ t := s \odot t$, for $s, t \in \mathcal{A}$, where the multiplication is given by (2). For the ground state and annihilation operators we assume that $\langle 0|0\rangle = 0$, $a_{s^*}|0\rangle = 0$, for $s^* \in \mathcal{A}^*$.The proper definition of action of annihilation operators on the whole algebra $\mathcal{A}$ is a problem. For the pairing $\langle -|-\rangle^n : \mathcal{A}^{*n} \otimes \mathcal{A}^n \longrightarrow \mathbb{C}$ we assume in addition that we have the following formulae

$$\langle 0|0\rangle^0 := 0, \quad \langle i|j\rangle^1 := (x^{*i}|x^j) = \delta^{ij}, \quad \langle s|t\rangle^n := \langle \tilde{s}|P_n\tilde{t}\rangle n_0 \quad \text{for} \quad n \neq 2 \quad (3)$$

where $\tilde{s}, , \tilde{t}, \in E^{\otimes n}$, $P_n : E^{\otimes n} \longrightarrow E^{\otimes n}$ is an additional linear operator and

$$\langle i_1 \cdots i_n|j_1 \cdots j_n\rangle_0^n := \langle i_1|j_1\rangle^1 \cdots \langle j_n|j_n\rangle^1.$$

Observe that we need two sets $\Pi := \{\Pi_n\}$ and $P := \{P_n\}$ of operators and the action

$$a : s^* \otimes t \in \mathcal{A}^{*k} \otimes \mathcal{A}^n \longrightarrow a_{s^*}t \in \mathcal{A}^{n-k}. \quad (4)$$

of annihilation operators for the algebraic description of our system. In this way the triple $\{\Pi, P, a\}$, where Π and P are set of linear operators and a is the action of annihilation operators, is the initial data for our model. The problem is to find and classify all triples of initial data which lead to the well–defined models. The general solution for this problem is not known for us. Hence we must restrict our attention for some examples.

If operators P and Π and the action a of annihilation operators are given in such a way that there is unique, nondegenerate, positive definite scalar product, creation operators are adjoint to annihilation ones and vice versa, then we say that we have a well–defined system with generalized statistics.

Example 1. We assume here that $\Pi_n \equiv P_n \equiv id_{E^{\otimes n}}$. This means that the algebra of states $\mathcal{A}$ is identical with the full tensor algebra TE over the space E, and the second algebra $\mathcal{A}^*$ is identical with the tensor algebra TE^*. The action (4) of annihilation operators is given by the formula $a_{x^{*i_k} \otimes \cdots \otimes x^{*i_1}}(x^{j_1} \otimes \cdots \otimes x^{j_n}) := \delta_{i_1}^{j_1} \cdots \delta_{i_k}^{j_k}\, x^{j_{n-k+1}} \otimes \cdots \otimes x^{j_n}$. For the scalar product we have the equation $\langle i_n \cdots i_1|j_1 \cdots j_n\rangle^n := \delta^{i_1 j_1} \cdots \delta^{i_n j_n}$. It is easy to see that we have the relation and $a_{x^{*i}} a_{x_j} := \delta_i^j \mathbf{1}$. In this way we obtain the most simple example of well–defined

system with generalized statistics. The corresponding statistics is the so–called infinite (Bolzman) statistics [8, 9].

Example 2: For this example we assume that $\Pi_n \equiv id_{E^{\otimes n}}$. This means that $\mathcal{A} \equiv TE$ and TE^*. For the scalar product and for the action of annihilation operators we assume that there is a linear and invertible operator $T : E^* \otimes E \longrightarrow E \otimes E^*$ defined by its matrix elements $T(x^{*i} \otimes x^j) = \Sigma_{k,*l}\, T^{*ij}_{k*l}\, x^k \otimes x^{*l}$, such that we have $(T^{*ij}_{k*l})^* = \overline{T}^{*ji}_{l*k}$, i.e. $T^* = \overline{T}^t$,and $(T^t)^{*ij}_{k*l} = T^{*ji}_{l*k}$. Note that this operator not need to be linear, one can also consider the case of nonlinear one. We also assume that the operator T^* act to the left, i.e. we have the relation

$$(x^{*j} \otimes x^i)T^* = \Sigma_{l,*k}\, (x^l \otimes x^{*k})\, \overline{T}^{*ji}_{l*k}, \tag{5}$$

and $(T(x^{*i} \otimes x^j))^* \equiv (x^{*j} \otimes x^i)T^*$.The operator T given by the formula (5) is said to be ***a twist*** or ***a cross*** operator. The operator T describes the cross statistics of quasiparticles and quasiholes. The set P of projectors is defined by induction $P_{n+1} := (id \otimes P_n) \circ R_{n+1}$, where $P_1 \equiv id$ and the operator R_n is given by the formula $R_n := id + \tilde{T}^{(1)} + \tilde{T}^{(1)}\tilde{T}^{(2)} + \cdots + \tilde{T}^{(1)} \ldots \tilde{T}^{(n-1)}$, where $\tilde{T}^{(i)} := id_E \otimes \cdots \otimes \tilde{T} \otimes \cdots \otimes id_E$, $\tilde{T}$ on the i–th place, and $(\tilde{T})^{ij}_{kl} = T^{*ki}_{l*j}$. If the operator $\tilde{T}$ is a bounded operator acting on some Hilbert space such that we have the following Yang-Baxter equation on $E \otimes E \otimes E$

$$(\tilde{T} \otimes id_E) \circ (id_E \otimes \tilde{T}) \circ (\tilde{T} \otimes id_E) = (id_E \otimes \tilde{T}) \circ (\tilde{T} \otimes id_E) \circ (id_E \otimes \tilde{T}), \tag{6}$$

and $||\tilde{T}, || \leq 1$, then according to Bo$\dot{z}$ejko and Speicher [14] there is a positive definite scalar product

$$\langle s|t\rangle^n_T := \langle s|P_n t\rangle^n_0 \tag{7}$$

for $s, t \in \mathcal{A}^n \equiv E^{\otimes n}$. Note that the existence of nontrivial kernel of operator $P_2 \equiv R_1 \equiv id_{E\otimes E} + \tilde{T}$ is essential for the nondegeneracy of the scalar product [37]. One can see that if this kernel is trivial, then we obtain well–defined system with generalized statistics [39, 40].

Example 3: If the kernel of P_2 is nontrivial, then the scalar product (7) is degenerate. Hence we must remove this degeneracy by factoring the mentioned above scalar product by the kernel. We assume that there is an ideal $I \subset TE$ generated by a subspace $I_2 \subset \ker P_2 \subset E \otimes E$ such that $a_{s^*} I \subset I$ for every $s^* \in \mathcal{A}^*$, and for the corresponding ideal $I^* \subset E^* \otimes E^*$ we have $a_{s^*} t = 0$ for every $t \in TE$ and $s^* \in I^*$. The above ideal I is said to be Wick ideal [37]. We have here the following formulae $\mathcal{A} := TE/I, \quad \mathcal{A}^* := TE^*$ for our algebras. The projection Π is the quotient map $\Pi : \tilde{s}, \in TE \longrightarrow s \in TE/I \equiv \mathcal{A}$. For the scalar

product we have here the following relation $\langle s|t\rangle_{B,T} := \langle\tilde{s}|\tilde{t}\rangle_T$ for $s = P_m(\tilde{s})$ and $t = P_n(\tilde{t})$. One can define here the action of annihilation operators in such a way that we obtain well–defined system with generalized statistics [40].

Example 4: If $B = \frac{1}{\mu}\tilde{T},$, where μ is a parameter, then the third condition (31) is equivalent to the well known Hecke condition for $\tilde{T}$, and we obtain the well–known relations for Hecke symmetry and quantum groups [15, 16, 55].

Physical effect. Let us consider the system equipped with generalized statistics and described by two operators T and B like in Example 4. We assume here in addition that a linear and Hermitian operator $S : E \otimes E \longrightarrow E \otimes E$ such that

$$S^{(1)}S^{(2)}S^{(1)} = S^{(2)}S^{(1)}S^{(2)}, \quad \text{and} \quad S^2 = id_{E\otimes E} \tag{8}$$

is given. If we have the following relation $\tilde{T}, \equiv B \equiv S$, then it is easy to see that the conditions (31) are satisfied and we have well–defined system with generalized statistics. Let us assume for simplicity that the operator S is diagonal and is given by the following equation $S(x^i \otimes x^j) = \epsilon^{ij} x^j \otimes x^i$, for $i, j = 1, \ldots, N$, where $\epsilon^{ij} \in \mathbb{C}$, and $\epsilon^{ij}\epsilon^{ji} = 1$. In general we have $c_{ij} = -(-1)^{\Sigma_{ij}} q^{\Omega_{ij}}$, where $\Sigma := (\Sigma_{ij})$ and $\Omega := (\Omega_{ij})$ are integer–valued matrices such that $\Sigma_{ij} = \Sigma_{ji}$ and $\Omega_{ij} = -\Omega_{ji}$. The algebra $\mathcal{A}$ is here a quadratic algebra generated by relations

$$x^i \odot x^j = \epsilon^{ij} x^j \odot x^i, \quad \text{and} \quad (x^i)^2 = 0 \quad \text{if} \quad \epsilon^{ii} = -1 \tag{9}$$

We also assume that $\epsilon^{ii} = -1$ for every $i = 1, \ldots, N$. In this case the algebra $\mathcal{A}$ is denoted by $\Lambda_\epsilon(N)$. Now let us study the algebra $\Lambda_\epsilon(2)$, where $\epsilon^{ii} = -1$ for $i = 1, 2$, and $\epsilon^{ij} = 1$ for $i \neq j$, in more details. In this case our algebra is generated by x^1 and x^2 such that we have $x^1 \odot x^2 = x^2 \odot x^1, \quad (x^1)^2 = (x^2)^2 = 0$. Note that the algebra $\Lambda_\epsilon(2)$ is an example of the so–called $Z_2 \oplus Z_2$–graded commutative color Lie superalgebra [56]. Such algebra can be transformed into the usual Grassmann algebra Λ_2 generated by Θ^1 and Θ^2 such that we have the anticommutation relation $\Theta^1\ \Theta^2 = -\Theta^2\ \Theta^1$, and $(\Theta^1)^2 = (\Theta^1)^2 = 0$. In order to do such transformation we use the Clifford algebra C_2 generated by e^1, e^2 such that we have the relations $e^i\ e^j + e^j\ e^i = 2\delta^{ij}$, for $i, j = 1, 2$. For generators x^1, and x^2 of the algebra $\Lambda_\epsilon(2)$ the transformation is given by $\Theta^1 := x^1 \otimes e^1$, for $\Theta^2 := x^2 \otimes e^2$. It is interesting that the algebra $\Lambda_\epsilon(2)$ can be represented by one Grassmann variable Θ, $\Theta^2 = 0$ as follows $x^1 = (\Theta, 1), \quad x^2 = (1, \Theta)$. For the product $x^1 \odot x^2$ we obtain $x^1 \odot x^2 = (\Theta, \Theta)$.

In physical interpretation generators Θ^1 and Θ^2 of the algebra Λ_2 represents two fermions. They anticommute and according to the Pauli exclusion principle

we can not put them into one energy level. Observe that the corresponding generators x^1 and x^2 of the algebra $\Lambda_\epsilon(2)$ commute, their squares disappear and they describe two different quasiparticles. This means that these quasiparticles behave partially like bosons, we can put them simultaneously into one energy levels, and single fermion can be transformed under certain interactions into a system of two different quasiparticles.

3 Interaction Processes and Entwining Structures [3]

Let us assume that an interacting system of charged particles and quanta is given as a starting point for the study of particle processes. Our main assumption is that there is a set of initial configurations of the system and there is also a set of final configurations representing possible results of interactions. An arbitrary initial configuration can be transform into final one as a result of interaction processes. Every such transformation is said to be a *process*. Such transformation can be done in several different way. All of them should produce the same result. Hence there is the uniqueness problem for such description. If all final configurations for the system under considerations can be described in an unique way as a result of transformation of an initial configuration, then we say that the system is equipped with a *coherent process*.

In this paper possible particle processes are studied in terms of monoidal categories. Our considerations are based on assumptions discussed in previous publications [3, 6, 48, 49] and shortly summarized in the Preliminaries. Particle systems equipped with possibility to absorption or emission of certain quanta are described in terms of a category of modules and comodules. The process of absorption of quanta of the external field can be described as a left or right coaction of $\mathcal{C}$ on $\mathcal{M}$. Dually, the process of emission of quanta can be described as a left or right action of $\mathcal{C}$ on $\mathcal{M}$. In this case the coherence means that left and right (co-)action leads to the equivalent result. In our approach the equivalence can be expressed by an invertible entwining map $\tau : \mathcal{C} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{C}$ [57]. Composite systems which contain both charged particles and quanta are described as entwined modules. It is shown that under some assumptions there is an algebra $\mathcal{W}(\mathcal{A}, \mathcal{C}, \tau)$ whose representations represent the corresponding quantum states.

Let $\mathfrak{C} = \mathfrak{C}(\otimes, k)$ be a monoidal category with duals. The monoidal operation $\otimes : \mathfrak{C} \times \mathfrak{C} \to \mathfrak{C}$ is a bifunctor which has a two-sided identity object k. The

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category $\mathfrak{C}$ can contain some special objects like algebras, coalgebras, modules or comodules, etc... According to our previous considerations all possible particle processes can be represented as arrows of the category $\mathfrak{C}$. In our case $k = \mathbb{C}$ is the field of complex numbers. If $f : \mathcal{U} \to \mathcal{V}$ is an arrow from $\mathcal{U}$ to $\mathcal{V}$, then the object $\mathcal{U}$ represents physical objects before interactions and $\mathcal{V}$ represents possible results of interactions. We assume that different objects of the underlying category describe physical objects of different nature, charged particles, quasiparticles or different species of quanta of an external field, etc... Let $\mathcal{U}$ be an object of the category $\mathfrak{C}$, then the object $\mathcal{U}^*$ corresponds for antiparticles, holes or quasiholes or dual field, respectively. If $\mathcal{U}$ and $\mathcal{V}$ are two different objects of the category $\mathcal{C}$, then the product $\mathcal{U} \otimes \mathcal{V}$ is also an object of the category, it represents a composite system composed from object of different nature.

Let $\mathcal{A}$ be an unital associative algebra, and $\mathcal{C}$ be a counital coassociative coalgebra in $\mathfrak{C}$. We use the notation $\Delta(f) := f^{(1)} \otimes f^{(2)}$ for the coalgebra comultiplication. In our approach charged particles are represented by the algebra $\mathcal{A}$, an external quantum field is characterized by coalgebra $\mathcal{C}$. Composite systems which contain both charged particles and quanta can be described as a product of copies of $\mathcal{A}$ and $\mathcal{C}$. The multiplication $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ represents the creation process of a single classical object from a composite system of objects of the same species. The result of comultiplication $\triangle : \mathcal{B} \to \mathcal{B} \otimes \mathcal{B}$ represents a 'composition' process from copies of objects of the same nature. Let $\mathcal{C}$ be a coalgebra and $\mathcal{C}^*$ be an algebra in duality, i. e. we have a bilinear pairing $\langle -, - \rangle : \mathcal{C} \otimes \mathcal{C}^* \to k$ such that $< \triangle f, s \otimes t >=< f, m(s \otimes t) >$, where $f \in \mathcal{C}$ and $s, t \in \mathcal{C}^*$. In our physical interpretation this duality indicates certain relations between processes.

Entwined structures. Let $\mathcal{A}$ be an algebra, and $\mathcal{C}$ be a coalgebra. We denote by $M_{\mathcal{A}}^{\mathcal{C}}$ the category of right $\mathcal{A}$-modules and right $\mathcal{C}$-comodules. A mapping $\tau : \mathcal{C} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{C}$ such that

$$\begin{array}{ll} \tau \circ (id \otimes m) = (m \otimes id) \circ \tau_{23} \circ \tau_{12}, & \tau \circ (f \otimes 1) = 1 \otimes f, \\ (id \otimes \Delta) \circ \tau = \tau_{12} \circ \tau_{23} \circ (\Delta \otimes id), & (id \otimes \varepsilon) \circ \tau = \varepsilon \otimes id, \end{array} \tag{10}$$

is said to be entwining, where $\tau_{12} := \tau \otimes id$, and $f \in \mathcal{C}$. Let $\mathcal{A}$ be a $\mathcal{C}$-Galois extension of $\mathcal{B}$, then there is a unique entwining map $\tau : \mathcal{C} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{C}$ (see [57]). The entwining map is given by $\tau(f \otimes a) := \beta \circ (id \otimes_{\mathcal{B}} m) \circ (\chi \otimes id)$, where $\chi := \beta^{-1}(1 \otimes f), f \in \mathcal{C}$. We use the following notation $\tau(f \otimes a) := a_{(1)} \otimes f_{(2)}$ for the entwining τ and $\mathcal{A}^n := \underbrace{\mathcal{A} \otimes \cdots \otimes \mathcal{A}}_{n}$ for tensor product of n copies of $\mathcal{A}$.

Let us define a mapping $\Psi_{1n} : \mathcal{C} \otimes \mathcal{A}^n \to \mathcal{A}^n \otimes \mathcal{C}$ by the relations $\Psi_{11} :=$

$\tau, \quad \Psi_{1n} := \underbrace{(id \otimes \tau) \circ \cdots \circ (\tau \otimes id)}_{n}$. If τ is invertible, then we also define $\Psi_{n1} := (\Psi_{1n})^{-1}$.

Entwined modules. A right $\mathcal{A}$-module $\mathcal{M}$ equipped with an action $\alpha : \mathcal{M} \otimes \mathcal{A} \to \mathcal{M}$ which is also a right $\mathcal{C}-$ comodule with a coaction $\delta : \mathcal{M} \to \mathcal{M} \otimes \mathcal{C}$ such that

$$\begin{array}{ccccccc} \mathcal{M} \otimes \mathcal{A} & \to & \to & \mathcal{M} & \to & \to & \mathcal{M} \otimes \mathcal{C} \\ \| & & & & & & \| \\ \mathcal{M} \otimes \mathcal{A} & \stackrel{\delta}{\to} & \mathcal{M} \otimes \mathcal{C} \otimes \mathcal{A} & \stackrel{id \otimes \tau}{\to} & \mathcal{M} \otimes \mathcal{A} \otimes \mathcal{C} & \stackrel{\alpha}{\to} & \mathcal{M} \otimes \mathcal{C} \end{array} \tag{11}$$

is said to be an entwined module or $(\mathcal{A}, \mathcal{C}, \tau)$-module. Let $\mathcal{M}$ and $\mathcal{N}$ be two $(\mathcal{A}, \mathcal{C}, \tau)$-modules. A mapping $f : \mathcal{M} \to \mathcal{N}$ such that

$$\begin{array}{ccccccc} \mathcal{A} \otimes \mathcal{M} & \stackrel{\alpha}{\to} & \mathcal{M} & & \mathcal{M} & \stackrel{\delta}{\to} & \mathcal{M} \otimes \mathcal{C} \\ id \otimes f \downarrow & & f \downarrow & & \downarrow f & & \downarrow f \otimes id \\ \mathcal{A} \otimes \mathcal{N} & \stackrel{\alpha}{\to} & \mathcal{N} & & \mathcal{N} & \stackrel{\delta}{\to} & \mathcal{N} \otimes \mathcal{C} \end{array} \tag{12}$$

is said to be a $(\mathcal{A}, \mathcal{C}, c)$-module morphism. It is obvious that the collection of all $(\mathcal{A}, \mathcal{C}, c)$-modules and $(\mathcal{A}, \mathcal{C}, c)$-module morphisms is a category. We denote this category by $\mathcal{M}(\mathcal{A}, \mathcal{C}, c)$. For an algebra $\mathcal{A}$, and a coalgebra $\mathcal{C}$ with a coaction $\delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{C}$ we define $\mathcal{C}^n_{\mathcal{A}} := \underbrace{\mathcal{C} \otimes \cdots \otimes \mathcal{C}}_{n} \otimes \mathcal{A}$, $n = 1, 2, \ldots$. The space $\mathcal{C}^n_{\mathcal{A}}$ is (i) a right $\mathcal{A}$-module, (ii) a right $\mathcal{C}$-comodule, and (iii) a $(\mathcal{A}, \mathcal{C}, \tau)$-module, $n = 1, 2, \ldots$.

We define the action $\alpha_n : \mathcal{C}^n_{\mathcal{A}} \otimes \mathcal{A} \to \mathcal{A}$ and the coaction $\delta_n : \mathcal{C}^n_{\mathcal{A}} \to \mathcal{C}^n_{\mathcal{A}} \otimes \mathcal{C}$ by the following formulae $\alpha_n := id \otimes m, \quad \delta_n := id \otimes \delta$. This simply follows from the associativity and coassociativity.

Crossed product. Let $\mathfrak{C} = (\mathfrak{C}_0, \mathfrak{C}_1)$ be a category whose objects $\mathfrak{C}_0$ are associative and unital algebras over a field k and whose morphisms are algebra morphisms. For our purposes here, we shall denote by $\mathcal{A} \otimes \mathcal{B}$ the tensor product $\mathcal{A} \otimes_k \mathcal{B}$, where $\mathcal{A}, \mathcal{B}$ are considered as k-linear spaces. A linear mapping $\Psi : \mathcal{B} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}$ such that we have the following relations

$$\Psi \circ (id_{\mathcal{B}} \otimes m_{\mathcal{A}}) = (m_{\mathcal{A}} \otimes id_{\mathcal{B}}) \circ (id_{\mathcal{A}} \otimes \Psi) \circ (\Psi \otimes id_{\mathcal{A}}), \tag{13}$$

$$\Psi \circ (m_{\mathcal{B}} \otimes id_{\mathcal{A}}) = (id_{\mathcal{A}} \otimes m_{\mathcal{B}}) \circ (\Psi \otimes id_{\mathcal{B}}) \circ (id_{\mathcal{B}} \otimes \Psi) \tag{14}$$

is said to be an algebra cross or twist [59]. We use here the notation $\Psi(b \otimes a) = \Sigma a_{(1)} \otimes b_{(2)}$ for $a \in \mathcal{A}, b \in \mathcal{B}$. The tensor product $\mathcal{A} \otimes_k \mathcal{B}$ of algebras $\mathcal{A}$ and $\mathcal{B}$

equipped with the multiplication

$$m_\Psi := (m_\mathcal{A} \otimes m_\mathcal{B}) \circ (id_\mathcal{A} \otimes \Psi \otimes id_\mathcal{B}) \tag{15}$$

is an associative algebra called a crossed product with respect to the cross symmetry Ψ [59] and it is denoted by $\mathcal{W} = \mathcal{W}_\Psi(\mathcal{A},\mathcal{B}) = \mathcal{A} \otimes_\Psi \mathcal{B}$. There is one to one correspondence between algebra cross $\Psi : \mathcal{B} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}$ and crossed product $\mathcal{W}$ of algebras $\mathcal{A}$ and $\mathcal{B}$. Let $(\mathcal{A},\mathcal{C},\tau)$ an entwining structure $(\mathcal{A},\mathcal{C},\tau)$, and $\mathcal{C}^* := \hom(\mathcal{C},k)$, be the dual of $\mathcal{C}$. Observe that there is the bilinear pairing $\langle -,- \rangle : \mathcal{C} \otimes \mathcal{C}^* \to k$ defined by $\langle f,x \rangle \equiv ev(f \otimes x) := f(x)$. If there exists a unique mapping $\widetilde{\tau} : \mathcal{A} \otimes \mathcal{C}^* \to \mathcal{C}^* \otimes \mathcal{A}$ such that

$$\begin{array}{ccc} \mathcal{C} \otimes \mathcal{A} \otimes \mathcal{C}^* & \stackrel{id\otimes\widetilde{\tau}}{\longrightarrow} & \mathcal{C} \otimes \mathcal{C}^* \otimes \mathcal{A} \\ \downarrow \tau \otimes id & & \downarrow ev \otimes id \\ \mathcal{A} \otimes \mathcal{C} \otimes \mathcal{C}^* & \stackrel{id\otimes ev}{\longrightarrow} & \mathcal{A}, \end{array} \tag{16}$$

then the entwining structure $(\mathcal{A},\mathcal{C},\tau)$ is said to be $\mathcal{C}^*$-factorizable. If $(\mathcal{A},\mathcal{C},\tau)$ a factorizable entwining structure, then the mapping $\widetilde{\tau} : \mathcal{A} \otimes \mathcal{C}^* \to \mathcal{C}^* \otimes \mathcal{A}$ is an algebra cross.

Let $(\mathcal{A},\mathcal{C},\tau)$ a factorizable entwining structure $(\mathcal{A},\mathcal{C},\tau)$, then there is a cross symmetry Ψ and the corresponding algebra cross tensor product $\mathcal{W}(\mathcal{A},\mathcal{C},\tau) = \mathcal{A} \otimes_\Psi \mathcal{C}^*$ of algebras $\mathcal{A}$ and $\mathcal{C}^*$. If $(\mathcal{A},\mathcal{C},\tau)$ a factorizable entwining structure, then according to the last lemma there exists a unique algebra cross $\widetilde{\tau} : \mathcal{A} \otimes \mathcal{C}^* \to \mathcal{C}^* \otimes \mathcal{A}$.

4 Particles interactions and quantum logic [4]

It is well-known that from algebraic point of view that q-deformed commutation relations for particles equipped with generalized statistics can be described in terms of the so-called Wick algebras [37]. The construction Wick algebras is based on a single operator T which describe the deformation. A second operator B is required for consistency relations. A proposal for the general algebraic formalism for description of particle systems equipped with an arbitrary generalized statistics based on the concept of monoidal categories with duality has been given by the Author in [6]. The physical interpretation for this formalism was shortly indicated. A few examples of applications for this formalism are considered in [48, 60].

[4]Last change January 28, 2003

In this paper we are going to study possible states of an interacting particle systems in terms of quantum logics. Our notion of quantum logics is specific and is partially based on the book [63]. All our considerations are based on previously developed algebraic formalism for particles with generalized statistics [5, 62]. We describe a system with generalized statistics as a quantum logics representation. Our study is motivated by possible physical applications to describe pseudoparticle configurations of magnetic crystal and related topics, see [61] and reference therein.

Discrete system and quantum logics. The starting point for our study is an interacting particle system. It is natural to assume that the whole physical world is divided into two parts: a classical system and its quantum environment. The classical system represents a physical reality, all that can be observed. The quantum environment represents all quantum possibilities to become a part of the reality in the future. Our fundamental assumption is that an initial particle configuration is transformed under interaction into a composite system consisting all possible results of interactions. We assume that all possible composite systems as a results of interactions can be constructed from an initial set of elementary ones. We would like to construct an algebraic model representing these possibilities [62].

A system which contains a charge and certain number of quanta as a result of interaction with the quantum field is said to be a *dressed particle* or a *lattice*. We describe a dressed particle as a non-local discrete system which contains n centers (vertexes or lattice sites). Every center can be dressed by quanta of N different sorts. A center with a place for certain quanta is said to be a *quasihole* or *an empty quantum level*. A centre dressed with a single quantum of the field is said to be a *quasiparticle*.

Let us denote by $L_n := (I_n, \mathbb{Q})$ a collection of all possible states corresponding to a dressed particle with n-centers and N-sorts quanta. We assume that this collection contains a finite collection of elementary (or primary) states $\mathbb{Q} \equiv \mathbb{Q}_N := \{1, 2, \ldots, N < \infty\}$, a corresponding collection of $*$-conjugated states $\mathbb{Q}^* \equiv \mathbb{Q}_N^* := \{*N, \ldots, *2, *1\}$ and a collection of maps $L_n := \{\sigma : k \in I_n \mapsto \sigma(k) := i_k \in \mathbb{Q}_N \cup \mathbb{Q}_N^* \cup \emptyset\}$, where $k = 1, \ldots, n$, $I_n := \{1, \ldots, n\}$; $n = 1, 2, \ldots,$ called composite states (or chain of states). This means that $\sigma \in L_n$ is a finite sequence of length n .i.e $\sigma := \{i_1, \ldots, i_n\}$, $i_k \in \mathbb{Q}_N \cup \mathbb{Q}_N^* \cup \emptyset$, $k = 1, \ldots, n$. The empty sequence is denoted by $\sigma(0) := \emptyset$. We assume that $L_0 := \emptyset$ for $n = 0$. There is a series of actions $\varrho_n : \pi \in S_n \to \varrho_n(\pi) \in End(L_n)$, where S_n is the

n-th symmetric group, and

$$\varrho(\pi) := \begin{pmatrix} \sigma \\ \sigma \circ \pi^{-1} \end{pmatrix}, \quad \sigma \in L, \ \pi \in S_n. \tag{17}$$

For $\sigma := \{i_1, \ldots, i_n\}$ we define an $*$-operation by the formula $*\sigma := \{*i_n, \ldots, *i_1\}$, where $*(i) := *i$, $*(*i) := **i := i$. Observe that 1) $\emptyset \in L$; 2) if $\sigma \in L_n$, then there is $*\sigma \in L_n$; 3) if $\sigma_n \in L_n$ for $n = 1, 2, \ldots$, then $\bigcup_{n=1,2,\ldots} \sigma_n \in L$; and $*(*\sigma) = \sigma, \ \sigma \cap *\sigma = \emptyset$. We say that $L := \bigcup_{n=0,1,\ldots} L_n$ is a family of quantum logics if and only if an transitive and anti-reflexive relation $\perp$ in L is defined. The relation $\perp$ is said to be an orthogonality, we denote by $\sigma \perp \sigma'$ a pair of orthogonal elements of L [63]. We assume in addition that $\{*i\} \perp \{j\}$, if $i \neq j$.

Let $\sigma, \sigma' \in L$ and $\sigma \perp \sigma'$, then the pair (σ, σ') is said to be a mutually excluded. A collection $\{\sigma_{i_1}, \ldots, \sigma_{i_n}\}$ is said to a complete if and only if $\sigma_{i_1} \perp \ldots \perp \sigma_{i_n} = \mathbb{Q} \cup \mathbb{Q}^*$. Obviously $\mathbb{Q} \cup \mathbb{Q}^*$ is complete $\{i_1\} \perp \ldots \perp \{i_N\} = \mathbb{Q} \cup \mathbb{Q}^*_N$.

Let L and L' be quantum logics. A mapping $f : L \to L'$ such that 1) $f(\emptyset) = \emptyset$; 2) $f(*\sigma) = *f(\sigma)$; 3) if $\sigma_{i_1} \perp \cdots \perp \sigma_{i_n}$, then $f(\sigma_{i_1}) \perp \cdots \perp f(\sigma_{i_n})$ is said to be a logic morphism.

Linear representations. A (symmetric) operad E is defined as a collection of sets $\mathbf{E} := \{E(k) : k = 0, 1, 2, \ldots, n, \ldots\}$, equipped with a collection of structure mappings

$$\gamma_{k_1 \cdots k_n} : E(n) \times E(k_1) \times \cdots \times E(k_n) \to E(k_1 + \cdots + k_n) \tag{18}$$

for every $k_1, \ldots, k_n = 1, 2, \ldots$, $n = 1, 2, \ldots$ satisfying some known compatibility condition for composition and symmetric group actions [64]. We use the notation $\gamma_{k_1 \cdots k_n}(v; v_1, \cdots, v_n) := v(v_1, \cdots, v_n)$, where $v \in E(n), v_1 \in E(k_1), \cdots, v_n \in E(k_n)$. We assume that there is an element $\mathbf{1} \in E(0)$ called the unit.

Let $L := (\mathbb{Q}, P)$ be a quantum logic. A representation of L In an operad $\mathbf{E}$ is a map $x : L \to \mathbf{E}$ such that

$$\begin{array}{ll} x(\sigma) \in E(n), & \text{for } \sigma := \{i_1, \ldots, i_n\} \\ x(\emptyset) := \mathbf{1}, & x(*\sigma) = (x(\sigma))^*; \end{array} \tag{19}$$

and $x(\sigma_{i_1}) \perp \cdots \perp x(\sigma_{i_n})$ for $\sigma_{i_1} \perp \cdots \perp \sigma_{i_n}$. For the action ϱ_n of the symmetric group we assume that $x(\varrho_n(\pi)\sigma) = \varrho_n^E \circ x(\sigma)$, where ϱ^E is the corresponding action in the operad.

Here we restrict our attention to the particular choice $E(n) := E^{\otimes n}$, $E(0) := \mathbf{1}I$, $E(1) := E$, where E is a linear space over a field I equipped with a basis

$S := \{x(\{i\}) := x^i : i = 1, \ldots, N\}$. We have

$$x(\bigcup_{i=1,2,\ldots} \sigma_i) := \bigotimes_{i=1,2,\ldots} x(\sigma_i). \tag{20}$$

For $\sigma := \{i_1, \ldots, i_n\} \in P$ we have an extension $x(\sigma) := x^{i_1} \otimes \cdots \otimes x^{i_n}$. Similarly $S^* := \{x(*\sigma_i) := x^{*i} : i = 1, \ldots, N\}$ forms a basis of the conjugate space E^*. The pairing $g_E : E^* \otimes E \to I$ and the corresponding scalar product is given by $g_E(x^{*i} \otimes x^j) \equiv (x^{*i}|x^j) = \langle x^i|x^j\rangle := \delta^{ij}$.

Let $T : E^* \otimes E \to E \otimes E^*$ be a linear and Hermitian operator with matrix elements

$$T(x^{*i} \otimes x^j) = \Sigma\, T^{ij}_{kl}\; x^k \otimes x^{*l}, \tag{21}$$

then the quotient $\mathcal{W}(T) = T(E \oplus E^*)/I_T$, where the ideal I_T is given by the relation

$$I_T := gen\{x^{*i} \otimes x^j - \Sigma\, T^{ij}_{kl}\; x^k \otimes x^{\star l} - (x^{\star i}|x^j)\} \tag{22}$$

is said to be Hermitian Wick algebra [37].

Algebra representation and cross product. Let L be a quantum logic, $\mathcal{A}$ be an an unital and associative algebra,

$$\mathcal{A} := \bigoplus_n \mathcal{A}^n, \tag{23}$$

equipped with an associative multiplication $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ then there is a representation $x : L \to \mathcal{A}$ of L, such that $x(\sigma) := m_n(x^{i_1} \otimes \cdots \otimes x^{i_n})$ for $\sigma := \{i_1, \ldots, i_n\} \in P$, and $m_n := \underbrace{m(id \otimes \cdots (id \otimes m) \cdots)}_{n}$. A pair of algebras $\mathcal{A}$ and $\mathcal{A}^*$ is said to be conjugated algebras if and only if there is an antilinear and involutive isomorphism $(-)^* : \mathcal{A} \to \mathcal{A}^*$, i. e. we have the relations $m_{\mathcal{A}^*}(b^* \otimes a^*) = (m_{\mathcal{A}}(a \otimes b))^*, \quad (a^*)^* = a$, where $a, b \in \mathcal{A}$ and a^*, b^* are their images under the isomorphism $(-)^*$.

Let $(\mathcal{A}, \mathcal{A}^*)$ be a pair of conjugate algebras. A linear mapping $\Psi : \mathcal{A}^* \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}^*$ such that $\Psi|_{E^* \otimes E} = T + g_E$, and we have the following relations [59, 65]

$$\begin{array}{l} \Psi \circ (id_{\mathcal{A}^*} \otimes m_{\mathcal{A}}) = (m_{\mathcal{A}} \otimes id_{\mathcal{A}^*}) \circ (id_{\mathcal{A}} \otimes \Psi) \circ (\Psi \otimes id_{\mathcal{A}}), \\ \Psi \circ (m_{\mathcal{A}^*} \otimes id_{\mathcal{A}}) = (id_{\mathcal{A}} \otimes m_{\mathcal{A}^*}) \circ (\Psi \otimes id_{\mathcal{A}^*}) \circ (id_{\mathcal{A}^*} \otimes \Psi) \end{array} \tag{24}$$

is said to be a cross symmetry or generalized braidings generated by T. We use here the notation $\Psi(b^* \otimes a) = \Sigma a_{(1)} \otimes b^*_{(2)}$ for $a \in \mathcal{A}, b^* \in \mathcal{A}^*$. The tensor product $\mathcal{A} \otimes \mathcal{A}^*$ equipped with the multiplication

$$m_\Psi := (m_{\mathcal{A}} \otimes m_{\mathcal{A}^*}) \circ (id_{\mathcal{A}} \otimes \Psi \otimes id_{\mathcal{A}^*}) \tag{25}$$

is an associative algebra isomorphic to the Hermitian Wick algebra [59] and it is denoted by $\mathcal{W} = \mathcal{W}_\Psi(\mathcal{A}) = \mathcal{A} >\!\!\lhd_\Psi, \mathcal{A}^*$. Let H be a linear space. We denote by $L(H)$ the algebra of linear operators acting on H.

Let $\mathcal{W} \equiv \mathcal{A} >\!\!\lhd_\Psi, \mathcal{A}^*$ be a Hermitian Wick algebra. If $\pi_\mathcal{A} : \mathcal{A} \to L(H)$ is a representation of the algebra $\mathcal{A}$, such that we have the relation

$$\begin{array}{l} (\pi_\mathcal{A}(b))^* \pi_\mathcal{A}(a) = \Sigma \pi_\mathcal{A}(a_{(1)}) \pi_{\mathcal{A}^*}(b^*_{(2)}), \\ \pi_{\mathcal{A}^*}(a^*) := (\pi_\mathcal{A}(a))^*, \end{array} \tag{26}$$

then there is a representation $\pi_\mathcal{W} : \mathcal{W} \to L(H)$ of the algebra $\mathcal{W}$ [59]. The relations (26) are said to be a commutation relation if there is a positive definite scalar product on $\mathcal{A}$ such that $\langle \pi_{\mathcal{A}^*}(x^*) f | g \rangle = \langle f | \pi_\mathcal{A}(x) g \rangle$. Note that if we use the notation $\pi_\mathcal{A}(x^i) \equiv a^+_{x^i}, \quad \pi_{\mathcal{A}^*}(x^{*i}) \equiv a_{x^{*i}}$, and the cross T is given by its matrix elements (21), then the commutation relations (26) can be given in the following form

$$a_{x^{*i}} a^+_{x^j} - T^{ij}_{kl} \; a^+_{x^l} a_{x^{*k}} = \delta^{ij} \mathbf{1}. \tag{27}$$

Creation and annihilation operators. Let us consider creation and annihilation operators (CAO) for our case. We introduce creation operators as the multiplication in $\mathcal{A}$ $a^+_{x^i} v := m(x^i \otimes v)$ for every $v \in \mathcal{A}$. We use here the following notation for our CAO $|i_1, \ldots, i_n\rangle := m_n(x^{i_1} \otimes \cdots \otimes x^{i_n})$ for state vectors and $a_{x^i} \equiv a^+_i$ for creation operators corresponding for generators of the algebra $\mathcal{A}$. We have for example

$$a^+_{j_1} \cdots a^+_{j_n} |0\rangle = |i_1, \ldots, i_n\rangle. \tag{28}$$

For annihilation operators we assume that $a_{x^{i*}}|0\rangle \equiv a_i|0\rangle = 0$ for every $x^{*i} \in S^*$ and $a_{s^*} v \in \mathcal{A}^{n-k}$, for $s^* \in \mathcal{A}^{*k}$.The proper action of annihilation operators on the whole algebra $\mathcal{A}$ is a problem.

If a representation x of quantum logic L in an algebra A is given in such a way that there is an unique, nondegenerate and positive definite scalar product then we say that A is the noncommutative Fock space. This means that our quantum logic are represented by a system with generalized statistics.

Example 1. Let L be a quantum logic, where the orthogonality relation $\perp$ holds for every pair (σ, σ') such that $*\sigma \neq \sigma'$. Here the algebra of states $\mathcal{A}$ is the full tensor algebra TE over the space E, and the conjugate algebra $\mathcal{A}^*$ is identical with the tensor algebra TE^*. If $T \equiv 0$ then we obtain the most simple example of well–defined system with generalized statistics. The corresponding statistics is the so–called infinite (Bolzman) statistics [8, 9].

Example 2 Let Ψ^T be a generalized braiding generated by an operator $T : E^* \otimes E \to E \otimes E^*$. This means that $\Psi^T : TE^* \otimes TE \to TE \otimes TE^*$ is defined as a set of mappings $\Psi_{k,l} : E^{*\otimes k} \otimes E^{*\otimes k} \to E^{\otimes l} \otimes E^{*\otimes k}$, where $\Psi_{1,1} \equiv R := T + g_E$, and

$$\begin{array}{l} \Psi_{1,l} := R_l^{(l)} \circ \cdots \circ R_l^{(1)}, \\ \Psi_{k,l} := (\Psi_{1,l})^{(1)} \circ \ldots \circ (\Psi_{1,l})^{(k)}, \end{array} \tag{29}$$

here $R_l^{(i)} : E_l^{(i)} \to E_l^{(i+1)}$, $E_l^{(i)} := E \otimes \ldots \otimes E^* \otimes E \otimes \ldots \otimes E$ ($l+1$-factors, E^* on the i-th place, $i \leq l$) is given by the relation $R_l^{(i)} := \underbrace{id_E \otimes \ldots \otimes R \otimes \ldots \otimes id_E}_{l\ times}$, where R is on the i-th place, $(\Psi_{1,l})^{(i)}$ is defined in similar way like $R^{(i)}$. We also introduce the operator $\tilde{T} : E \otimes E \to E \otimes E$ by its matrix elements $(\tilde{T})^{ij}_{kl} = T^{ki}_{lj}$. If the operator $\tilde{T}$ is a bounded operator acting on some Hilbert space such that we have the following Yang-Baxter equation on $E \otimes E \otimes E$

$$(\tilde{T} \otimes id_E) \circ (id_E \otimes \tilde{T}) \circ (\tilde{T}, \otimes id_E) = (id_E \otimes \tilde{T}) \circ (\tilde{T} \otimes id_E) \circ (id_E \otimes \tilde{T}), \tag{30}$$

and $||\tilde{T}|| \leq 1$, then according to Bożejko and Speicher [14] there is a positive definite scalar product. Note that the existence of nontrivial kernel of operator $P_2 \equiv id_{E\otimes E} + \tilde{T}$ is essential for the nondegeneracy of the scalar product [37]. One can see that if this kernel is trivial, then we obtain the well–defined system with generalized statistics [38, 40].

Example 3: If a linear and invertible operator $B : E \otimes E \to E \otimes E$ defined by its matrix elements $B(x^i \otimes x^j) := B^{ij}_{kl}(x^k \otimes x^l)$ is given such that we have the following conditions

$$\begin{array}{c} B^{(1)}B^{(2)}B^{(1)} = B^{(2)}B^{(1)}B^{(2)}, \\ B^{(1)}T^{(2)}T^{(1)} = T^{(2)}T^{(1)}B^{(2)}, \\ (id_{E\otimes E} + \tilde{T})(id_{E\otimes E} - B) = 0, \end{array} \tag{31}$$

then $I := gen\{id_{E\otimes E} - B\}$ and one can prove that the corresponding system is well defined [38, 40]. In this case

$$\varrho^E_n(\tau) := B, \tag{32}$$

where τ is the transposition $(1, 2) \to (2, 1)$ in S_2.

5 Operads and Fock Spaces [5]

Let $\mathcal{M}(\otimes, \mathbb{C})$ be a monoidal category equipped with a monoidal operation (a bi-functor) $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$, and with the field $\mathbb{C}$ as the unit object. If there is also a $*$-operation $(-)^* : \mathcal{M} \to \mathcal{M}$ and pairing $g = \{g_U : U^* \otimes U \to \mathbb{C}\}$ satisfying some known axioms, then we say that we have a category with duality. One can also include cross or braids, see [Marcinek 1996] for more details.

We would like to construct a new category $\mathcal{P} = \mathcal{P}(I, \mathbf{0}, \odot, \oplus)$ whose objects are isomorphic classes of of objects of the initial category $\mathcal{M}$ and equipped with two special objects $I, \mathbf{0}$, a product $\otimes$ and coproduct $\oplus$ satisfying the distributivity and associativity conditions

$$\begin{array}{ll} (\mathcal{U} \odot \mathcal{V}) \odot \mathcal{W} \simeq \mathcal{U} \odot (\mathcal{V} \odot \mathcal{W}) & \mathcal{U} \oplus \mathbf{0} \simeq \mathcal{U} \\ (\mathcal{U} \oplus \mathcal{V}) \oplus \mathcal{W} \simeq \mathcal{U} \oplus (\mathcal{V}) \oplus \mathcal{W}) & \mathcal{U} \odot I \simeq \mathcal{U} \\ \mathcal{U} \odot (\mathcal{V} \oplus \mathcal{W}) \simeq (\mathcal{U} \odot \mathcal{V}) \oplus (\mathcal{U} \odot \mathcal{W}) & \mathcal{U} \oplus \mathcal{V} \simeq \mathcal{V} \oplus \mathcal{U} \end{array} \tag{33}$$

up to natural isomorphisms. We have the following assumptions for our operadicc description of noncommutative Fock space: 1) $\mathbf{0}$ is the empty class of objects; 2) $I = \mathbf{1}\mathbb{C}$ is the class corresponding to the unit object; 3) there is an ordered (finite) collection of classes $S := \{x^i : i = 1, \ldots, N < \infty\}$. These classes form a basis for a finite linear space E over a field of complex numbers $\mathbb{C}$. There is an ordered (finite) collection of conjugated classes $S^* := \{x^{*i} : i = N, N-1, \ldots, 1\}$. They form a basis for the complex conjugate space E^*. The pairing $(= |-) : E^* \otimes E \longrightarrow Iq$ is given by $(x^{*i}|x^j) := \delta^{ij}$. We assume that there is a set of linear projectors $\odot_n : E^{\otimes n} \longrightarrow E^{\otimes n}$, which is defined uniquely by an iteration procedure

$$\odot_{n+1} = \odot \circ (id \otimes \odot_n) = \odot \circ (\odot_n \otimes id), \tag{34}$$

where $\odot_1 := id$, and $\odot_2 := \odot$ is given as a data for our construction. Observe that the definition is unique if and only if our operation is associative. We are going to describe a collection of spaces $\mathbb{E} := \{\mathbb{E}(n) : n = 0, 1, 2, \cdots\}$, where $\mathbb{E}(0) = I$, $\mathbb{E}(1) = E$, and $\mathbb{E}(n) := Im \odot_n$ for $n > 1$. In this way we obtain a collection of mappings $\odot_n : E^{\otimes n} \to \mathbb{E}(n)$, where we use the notation $\odot_n(x^{i_1} \otimes \ldots \otimes x^{i_n}) := x^{i_1} \odot \ldots \odot x^{i_n}$. We assume that the set $\{\odot_n(x^{i_1}, \ldots, x^{i_n}) := x^{i_1} \odot \ldots \odot x^{i_n} : \sigma = (i_1, \ldots i_n) \in \Sigma\}$ forms a basis of $\mathbb{E}(n)$.

Composition operad. There is a collection of spaces $\mathbb{E} := \{\mathbb{E}(n) : n = 0, 1, 2, \cdots\}$, where $\mathbb{E}(0) = I$, $\mathbb{E}(1) = E$, and we have the following structure

[5] Last change May 19, 2003

map

$$\gamma_{\mathbb{E}} : \mathbb{E}(l) \times \mathbb{E}(n_1) \times \cdots \times \mathbb{E}(n_l) \to \mathbb{E}(n_1 + \cdots + n_l) \tag{35}$$

for every $n_1, \ldots, n_l = 1, 2, \ldots$, $l = 1, 2, \ldots$ such that they form a nonsymmetric operad. Here we use the following notation $\gamma_{\mathbb{E}}(v; v_1, \cdots, v_l) := v(v_1, \cdots, v_l)$, where $v \in \mathbb{E}(l), v_1 \in \mathbb{E}(n_1), \cdots, v_l \in \mathbb{E}(n_l)$.

Coproduct cooperad. There is the corresponding cooperad $\mathbb{E}^* := \{\mathbb{E}^*(n) : n = 0, 1, 2, \cdots\}$, where $\mathbb{E}^*(0) = \mathbb{C}\mathbf{1}^*$, $\mathbb{E}^*(1) = E^*$, and with the following structure map

$$\gamma_{\mathbb{E}^*} : \mathbb{E}^*(l) \times \mathbb{E}^*(n_1) \times \cdots \times \mathbb{E}^*(n_l) \to \mathbb{E}^*(n_1 + \cdots + n_l) \tag{36}$$

for every $n_1, \ldots, n_l = 1, 2, \ldots$, $l = 1, 2, \ldots$. We define creation operators $a^+_{x^i} v := \odot_{n+1}(x^i \otimes v)$, where $x^i \in E$, $v \in \mathbb{E}(n)$, and annihilation ones $a^-_{x^{*i}}\, x^{j_1} \odot \ldots \odot x^{j_n} := (x^{*i} \mid x^{j_1})_1\, x^{j_2} \odot \ldots \odot x^{j_n}$. We have here the following relations

$$a^-_{*i} \circ a^+_j = (x^{*i} \mid x^j)_1. \tag{37}$$

For every x^{*i} we define an operator $\iota_{x^{*i}}$ by the formula $\iota_{x^{*i}} x^j := (x^{*i} \mid x^j)_1$. We use the notation

$$\iota^{(k)}_{x^{*i}} x^{j_1} \odot \ldots \odot x^{j_k} \odot \ldots \odot x^{j_n} = x^{j_1} \odot \ldots \odot \underbrace{\iota_{x^{*i}}}_{k} x^{j_k} \odot \ldots \odot x^{j_n}. \tag{38}$$

We introduce new operators

$$b^+_i := a^+_i,$$
$$b^-_{*i}(x^{j_1} \odot \ldots \odot x^{j_k} \odot \ldots \odot x^{i_n}) = \sum_{k=1}^n \left(\iota^{(k)}_{x^{*i}} \circ \mathbb{T}(k)\right) x^{*i} \odot x^{j_1} \odot \ldots \odot x^{j_k} \odot \ldots \odot x^{j_n}. \tag{39}$$

We obtain here the following relations

$$b^-_{*i} \circ b^+_i - \sum_{k,l} T^{ij}_{kl}\ b^+_k b^-_{*l} = (x^{*i} \mid x^j), \tag{40}$$

where T^{ij}_{kl} are matrix elements $\mathbb{T}(1)(x^{*i} \otimes x^j) := \sum_{k,l} T^{ij}_{kl} x^k \otimes x^{*l}$. Paring operad is a collection of pairings $(-|-) := \{(-|-)_n : n = 1, 2, \ldots\}$, where $(-|-)_0 := 0$, $(-|-)_1$ is given above. Wick ordering operad is a collection of operators $\{\mathbb{T}(n) : \mathbb{E}^*(1) \otimes \mathbb{E}(n) \to \mathbb{E}(n) \otimes \mathbb{E}^*(1)\}$.

Fock operad representation. There are two familes of operators

$$a^+ := \{a^+_i : \mathbb{E}(n) \to \mathbb{E}(n+1), i = 1, \ldots, N\}, \tag{41}$$

and

$$a^- := \{a^-_{*i} : \mathbb{E}(n) \to \mathbb{E}(n-1), *i = N, N-1, \ldots, 1\}. \tag{42}$$

Noncommutative Fock space. If there is a set of nondegenerate and positive definite scalar products $\langle -|-\rangle := \{\langle -|-\rangle_n : n = 1, 2, \ldots\}$ such that $\langle 0|0\rangle = 0$, $\langle x^i|x^j\rangle_1 := (x^{*i}|x^j)$, $\langle u|v\rangle_n := (u^*|v)_n$ and

$$a^-_{x^{*i}}|0\rangle = 0, \quad \langle a^+_i v|w\rangle = \langle v|a^-_{*i} w\rangle, \tag{43}$$

then we say that our quantum system is well defined.

Commutation relations. Let us describe a series of representations

$$\varrho_n : \pi \in S_n \mapsto S_\pi \in \mathrm{End}(E^{\otimes n}), \tag{44}$$

where S_n is the symmetric group, and $S_\pi := S_{\tau_{i_1}} \circ \ldots S_{\tau_{i_k}}$, corresponds for the following decomposition of π on transpositions $\pi := \tau_{i_1} \circ \ldots \circ \tau_{i_k}$. If $\odot_n$ is a composition mapping and ϱ_n is a representation of the symmetric group S_n described above, then for every permutation $\pi \in S_n$ there is a new composition mapping $\odot_n \circ S_\pi$.

6 Particle processes and cobordisms with trees [6]

Let $\mathcal{C} := \{\Sigma\}$ be a collection of d–dimensional compact, oriented and smooth manifolds without boundary. All these manifolds can be multiconnected in general. If $\Sigma_1, \Sigma_2 \in \mathcal{C}$, then we denote by $\Sigma_1 \cup \Sigma_2$ their disjoint sum. For every manifold $\Sigma \in \mathcal{C}$ there is the corresponding manifold Σ^* such that $\partial(\Sigma \times [0,1]) = \Sigma \cup \Sigma^*$. We also assume for our study that two finite set $\Gamma := \{\mathbf{i}_1, \cdots, \mathbf{i}_N\}$ and $\Gamma^* := \{\mathbf{i}^*_N, \cdots, \mathbf{i}^*_1\}$ are given.

Let us assume that there is a corresponding collection $\mathcal{C}(\mathcal{Q}) := \{(\Sigma, \mathcal{Q}) : \Sigma \in \mathcal{C}, \mathcal{Q} \in \mathcal{Q}(\Gamma)\}$ of manifolds with discrete structure $\mathcal{Q}(\Gamma) := \{\mathcal{Q}(n) : n = 0, 1, \ldots\}$ on these manifolds, where $\mathcal{Q}(n)$ is a finite set of points of Σ with labels in Γ, $\mathcal{Q}(0) = \emptyset$. The corresponding collection $\mathcal{C}(\mathcal{Q}^*) := \{(\Sigma^*, \mathcal{Q}^*) : \Sigma^* \in \mathcal{C}, \mathcal{Q}^* \in \mathcal{Q}(\Gamma^*)\}$ is defined in a similar way. In our approach particle processes of interactions are described by pairs of the form $(\mathcal{M}, G)$ which transforms the initial configuration $(\Sigma_0, \mathcal{Q}_0)$ into the outgoing one $(\Sigma_1, \mathcal{Q}_1)$ representing the results of interactions. Here $\mathcal{M} : \Sigma_0 \longrightarrow \Sigma_1$ transforms the manifold Σ_0 into Σ_1, and $G : \mathcal{Q}_0 \longrightarrow \mathcal{Q}_1$ transforms the corresponding discrete structures. If $(\mathcal{N}, G')$ is the

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second pair, then we assume that there is a pair $(\mathcal{M} \circ \mathcal{N}, G \circ G')$, the composition of $(\mathcal{M}, G)$ and $(\mathcal{N}, G')$. We can use the concept of cobordisms manifold and rooted trees for the description of these mappings and their compositions. We describe here an arbitrary particle process as a cobordism manifold with a tree structure. Denote by $\mathcal{E}$ a collection of $d+1$–dimensional compact, oriented and smooth manifolds with boundary. We assume that for every manifold $\mathcal{M} \in \mathcal{E}$ with boundary $\partial\mathcal{M}$ there are two manifolds Σ_0 and Σ_1 in $\mathcal{C}$ such that the boundary $\partial\mathcal{M}$ is diffeomorphic to $\Sigma_0 \cup \Sigma_1^*$.

A $d+1$ manifold $\mathcal{M} \in \mathcal{E}$ with boundary $\partial\mathcal{M}$ such that there are two smooth diffeomorphisms $f_0 : \Sigma_0^* \longrightarrow \partial\mathcal{M}$, and $f_1 : \Sigma_1 \longrightarrow \partial\mathcal{M}$ is said to be a cobordism of Σ_0 and Σ_1 and it is denoted by $\mathcal{M}(f_0, f_1)$. Two cobordisms $\mathcal{M}(f_0, f_1)$ and $\mathcal{M}'(f_0', f_1')$ are said to be equivalent if there is a diffeomorphism $F : \mathcal{M} \longrightarrow \mathcal{M}'$ such that $f_0' = F f_0$ and $f_1' = F f_1$. The equivalence class of cobordisms of Σ_0 and Σ_1 up to diffeomorphisms is denoted by $\mathcal{M}(\Sigma_0, \Sigma_1)$. The collection of all classes cobordisms of Σ_0 and Σ_1 is denoted by $\mathcal{E}(\Sigma_0, \Sigma_1)$. Let $\mathcal{N}$ be a cobordism of Σ_2 and Σ_3 with diffeomorphisms $f_0' : \Sigma_2^* \longrightarrow \partial\mathcal{N}$ and $f_1' : \Sigma_3 \longrightarrow \partial\mathcal{N}$. In certain cases we can glue two cobordisms $\mathcal{M}$ and $\mathcal{N}$ along Σ_1 and Σ_2 by identifying the part of boundary of $\mathcal{M}$ diffeomorphic to Σ_1 with the part of $\partial\mathcal{N}$ diffeomrphic to Σ_2^*, respectively. The composition $(f_0')^{-1} f_1$ is then a diffeomorphism of Σ_1 onto Σ_2^*. In this way we obtain the cobordism of Σ_0 and Σ_2. We denote it by $\mathcal{M} \circ \mathcal{N}$. The operation of gluing of cobordisms up to diffeomorphisms define the following composition $\circ : \mathcal{E}(\Sigma_0, \Sigma_1) \times \mathcal{E}(\Sigma_1, \Sigma_2) \longrightarrow \mathcal{E}(\Sigma_0, \Sigma_2)$. One can glue three or more cobordimsms. One can see that these gluings define a semigroups structure on the collection of all cobordism for certain class of manifolds.

It is interesting to study cobordisms of manifolds with certain additional structures. If for example these manifolds are configuration spaces for multiparticle system, then we must restrict our gluings for those which are admissible for our physical problem. For this goal we need some additional assumptins. A *rooted tree* is a finite, loop free, connected graph which contains edges and nodes. There is one distinguisched edge called a root. Every node is inner and there are no outer nodes. A rooted tree with one node and n entrance edgees is said to be a *the prime n–tree*. In our physical interpretation entrance edges describe initial configurations of particles. The root is said to be an exit. It represents the unique final configuration. There also the corresponding concept of co–rooted trees. Starting with a set of prime 2–trees and 2–co–trees one can construct a graph G with arbitrary number of entrance and outgoing edges. Such graph can represent arbitrary particle processes. If we embed trees into the cobordism manifold, then we obtain cobordisms with a tree structure. More precisely, let $\mathcal{M}(\Sigma_0, \Sigma_1)$ be a

cobordism of Σ_0 and Σ_1 with discrete structures $\mathcal{Q}_0 := \mathcal{Q}(n)$ and $\mathcal{Q}_1 := \mathcal{Q}(m)$, respevtively, and let G be a graph with n entrance edges and m outgoing ones, then for the embeding of G in $\mathcal{M}$ we must connect n entrance edges of G with n points of $\mathcal{Q}_n(\Sigma_0)$, and m outgoing edges with m points of $\mathcal{Q}_1$. Note that we need here some additional restrictions for the equivalence of cobordisms. If the disrete structure represents fermions and bosons, then we need certain supermanifolds or some generalization for to describe the corresponding cobordisms. We can use here the notion of the so–called semisupermanifolds introduced by S. Duplij [68].

We denote by $Cob = Cob_d(\mathcal{Q})$ the category of cobordisms with a tree structure. Objects of this category are d–dimensional compact, oriented and smooth manifolds Σ equipped with a discrete structure $\mathcal{Q}(\Sigma)$. Morphisms are cobordism manifold with a tree structure. Composition of morphisms can be expressed as a gluings of cobordisms with trees.

Let us assume that we have the so–called multiparticle states operad $\Lambda := \{\Lambda(n) : n = 0, 1, 2, \cdots\}$, where $\Lambda(n)$ is an n–particle space of states, $\Lambda(0) = \mathbb{C}\mathbf{1}$, $\Lambda(1)$ is equipped with a basis with N elements $x^1, \cdots, x^N$, $\Lambda(n)$ for $n \geq 2$ can be obtained from Λ_1 by an n-ary operation $\circ_n : \Lambda \times \cdots \times \Lambda \longrightarrow \Lambda(n)$. If the n-ary operation can be obtained uniquely from a binary one $F \equiv F_2 : \Lambda \times \Lambda \longrightarrow \Lambda(2)$ by an iteration procedure, then we say that the configuration operad is well defined. One can introduce a category $Part \equiv Part(\Lambda)$ related to the operad Λ. Let us consider an arbitrary functor $\mathcal{Z} : Cob \longrightarrow Part$. Let $\mathcal{M}$ be a cobordism of Σ_0 and Σ_1, then the our goal is the construction of $\mathcal{Z}(\Sigma_0)$, $\mathcal{Z}(\Sigma_1)$, and $\mathcal{Z}(\mathcal{M})$ as an mapping $\Phi_{\mathcal{M}} : \mathcal{Z}(\Sigma_0) \longrightarrow \mathcal{Z}(\Sigma_1)$. We express Φ as the path integral

$$(\Phi_{\mathcal{M}}\mathcal{W})(\varphi) = \int \exp(-S(\varphi))\mathcal{W}(\varphi)D\varphi, \tag{45}$$

where φ is a field on $\mathcal{M}$ with a given boundary conditions on Σ_0 and Σ_1, S is a given action and $\mathcal{W}$ is an observable. Note that $\mathcal{Z}(\Sigma_k)$ should be expressed as a sequence $\Lambda(n_1) \times \cdots \times \Lambda(n_r)$, where $\Sigma_{i=1}^r n_i = n$ is the number of point particles represented by $\mathcal{Q}(\Sigma_k)$.

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